Convergence Analysis of Classes of Asymmetric Networks of Cucker–Smale Type With Deterministic Perturbations

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***Abstract*—We discuss two extensions of the Cucker-Smale flocking model with asymmetric coupling weights. The first model assumes a finite collection of autonomous agents aiming to perform a consensus process in the presence of identical internal dynamics. The second model describes a similar population of agents that perform velocity alignment with the restriction of collision-free orbits. Although qualitatively different, we explain how these models can be analyzed under a common framework. Rigorous analysis is conducted toward establishing sufficient conditions for asymptotic flocking to a synchronized motion. Applications of our results are compared with simulations to illustrate the effectiveness of our theoretical estimates.**

***Index Terms*—Convergence, networked control systems, nonlinear systems.**

I. Introduction

FOR THE past two decades, there has been broad interest in the study of cooperative dynamic algorithms that run among autonomous interconnected entities. Perhaps the most prominent family is this of consensus networks. The standard setting regards $n < \infty$ agents each of which is represented by a state $z_i \in \mathbb{R}$ for $i = 1, \dots, n$. The rate of change of the states $z_i,\ i = 1, \dots, n$ is governed by the following protocol:

$$\dot{z}_i = \sum_{j=1}^{n} w_{ij}(t)(z_j - z_i). \tag{1}$$

The non-negative numbers w_{ij} are coupling weights that characterize the effect of agent j onto the agent i. Certain criteria imposed on w_{ij} ensure the asymptotic consensus property

$$z_i(t) \to z^* \text{ as } t \to \infty \text{ for } i = 1, \dots, n.$$

The common equilibrium point $z^* \in \mathbb{R}$ lies in the set defined by the convex hull of the initial values. The research on connectivity conditions for (1) is a saturated subject (see [10], [11], [17], [19], [20], [31], and references therein). On the contrary, there are fewer works in nonlinear consensus dynamics, some of which we review below.

A. Types of Nonlinear Consensus Protocols

Manfredi and Angeli [13] and Moreau [18] study necessary and sufficient conditions of convergence to consensus in discrete and continuous time versions of nonlinear networked cooperative systems represented by

$$x_i(k+1) = f_i(\,k, x_1(k), \dots, x_n(k)\,), \quad k \in \mathbb{N},$$

and

$$\dot{x}_i = f_i(\,t, x_1, \dots, x_n\,), \quad t \in \mathbb{R},$$

respectively. Nonlinear versions of (1) were investigated in [1] and [22]

$$\dot{z}_i = \sum_{j=1}^{n} w_{ij}(t, z_j - z_i) \tag{1.2}$$

and

$$\dot{z}_i = \sum_{j=1}^{n} w_{ij}(t, z_j) - \sum_{j=1}^{n} w_{ij}(t, z_i). \tag{1.3}$$

Cucker and Smale [5], [6] introduce a system that models the emergence of bird flocking. In this setting, the state of agent i consist of its position $x_i \in \mathbb{R}^r$ and velocity $v_i \in \mathbb{R}^r$, stacked as $(x_i, v_i) \in \mathbb{R}^r \times \mathbb{R}^r$. For an initial configuration (x_i^0, v_i^0) for $i = 1, \dots, n$, the dynamics of the network are given by

$$\begin{aligned} \dot{x}_i &= v_i \\ \dot{v}_i &= \sum_{j=1}^{n} w_{ij}(x)(v_j - v_i). \end{aligned} \tag{2}$$

The class of dynamical networks that (2) belongs to is known as the second-order consensus class. Linear versions of such

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schemes have also been previously studied (see, for example, [15] and [32]). In (2), the coupling weights are assumed to be decreasing functions of distance in the following form:

$$w_{ij}(x) = \frac{K}{(\sigma^2 + \|x_i - x_j\|^2)^\beta} \tag{3}$$

where $K, \sigma, \beta > 0$ are coupling parameters and $\|.\|$ is the Euclidean norm. The form of w_{ij} imposes the following spatial decaying condition: the larger the relative distance between two agents, the smaller their interaction. It may then occur that the agents will not be positioning themselves sufficiently close. This may, in turn, result in the network failing to preserve strong enough connectivity to achieve global speed alignment. The objective in (2) is to derive initial conditions that ensure a strong enough network to allow global flocking to a common consensus state. The authors exploit the explicit form of w_{ij}, in particular, its symmetry, and use algebraic graph theory to establish a stability condition that involves the initial data and the parameters K, σ, and β from (3). Since the seminal work of Cucker and Smale, (2) has been improved in several ways. The work in [8] provides a simple proof for general vanishing symmetric couplings. In [16], Martin *et al.* derive convergence results for general symmetric weights without the infinite distance connectivity condition. In [23], the results are extended to asymmetric couplings. Somarakis and Baras [25], [27] provide extensions to asymmetric couplings with simple and switching connectivity in the presence of time delay. Furthermore, [4] examines (2) under the effect of repelling symmetric functions. A convergence result that summarizes the contribution of most of the aforementioned works illustrates the kind of sufficient conditions for asymptotic velocity alignment so that the network does not get dissolved.

Theorem 1 ([23], [27]): Suppose that the nonlinear dynamical network (2) with coupling weights $w_{ij}(x) \geq \psi(\max_{ij} |x_i - x_j|)$, where ψ is a non-negative integrable function. If the initial data satisfy

$$\max_{i,j} |v_i^0 - v_j^0| < \int_{\max_{i,j} |x_i^0 - x_j^0|}^{\infty} \psi(s)\, ds$$

then the solution (x, v) of (2) satisfies

1) $|v_i(t) - v_j(t)| \to 0$ as $t \to \infty$,
2) $\sup_{t\geq 0} |x_i(t) - x_j(t)| < \infty$, for $i, j = 1, \ldots, n$.

The underlying argument for the inequality condition involves a rate of convergence estimate of the state of the network to its equilibrium that explicitly depends on the network parameters.

For symmetric couplings, the appropriate theoretical machinery is provided through algebraic graph theory [3]. Otherwise, one should leverage results from non-negative matrix theory [9], [24]. All of the aforementioned references focus exclusively on system (2). The stability conditions of Theorem 1 shed no light in the event that the agents are expected to collectively execute a more complex coordination task, different than simple convergence to a common equilibrium.

B. Our Contributions

In this paper, we introduce and discuss a class of extensions of the classic Cucker–Smale model represented by (2) that considers nonlinear deterministic perturbations. These models describe how a group of agents can benefit from asymmetric state-dependent graphs with spatially decaying couplings to perform more elaborate collaborative tasks, rather than performing the simple task of converging to a consensus state.

The importance of considering perturbed versions of (2) is to understand the inherent interplay between being able to accomplish complex collective behaviors and the coupling structure of the underlying dynamical network. Our goal is to investigate to what extent we can push the envelope to modify existing gold-standard and well-studied dynamical networks to allow them to exhibit more complex collective behaviors. Toward this end, we address two types of state-dependent perturbations, relevant to network models of type (2) that describe real-world paradigms.

The first extension takes into account the scenario of agents aiming to perform a consensus process in the presence of an internal generic dynamic rule. For instance, birds in a flock can fly individually or in coordination with other birds, possibly toward a synchronized complex motion different than a simple velocity coordination. This feature is modeled with an extra nonlinear term in (2) that makes the inequality flocking condition of Theorem 1 inapplicable. In Section III-A, it is shown how this class of networks can be analyzed.

The second extension is a perturbation of (2) that destabilizes the coordination process so long as the relative position of agents falls below a certain safety margin. Such a safety restriction can prevent a network with fast decaying coupling strength from achieving global convergence. Similar to the first case, a generalization of Theorem 1 is needed to ensure flocking conditions. In Section III-B, we show that under a new set of technical conditions, design of flocking without collision can be achieved.

In addition, several examples are discussed to truthfully verify the strength of the imposed conditions in either case. This paper concludes with discussing extensions of our results for networks with nonlinear coupling terms.

C. Literature Review

The subject of multi-agent synchronization is rather mature and well documented in the context of linear networks (see [7] and [33] for control theoretic perspective or [21] and [34] for a physics approach and references therein). In [28], Somarakis *et al.* analyze first-order synchronization schemes with time-dependent topologies where the internal dynamics may destabilize the alignment process. The work establishes sufficient conditions between the internal rule and the coupling scheme, in order for asymptotic synchronization to occur. To the best of our knowledge, the literature lacks a study of synchronization networks where the state of the network is correlated to the strength of the coupling as such is the case in networks of Cucker–Smale type.

Prior works on the subject of flocking with collision avoidance in Cucker–Smale type networks include [4] and [26]. In [4], Cucker and Dong study a collision-avoidance problem (model (12) of our work) on a framework based on the symmetry assumption of coupling weights. Their results seize to hold when the couplings and the repelling functions are not symmetric. The asymmetry on the coupling terms has been highlighted in the literature as a very important feature in real-world biological and natural networks [2]. This paper is an outgrowth of [29] in various directions. The synchronization scheme is considered under milder assumptions and multidimensional internal dynamics. This gives rise to stability conditions that generalize the ones in [29]. Finally, for the collision-free model, we provide a detailed proof of the convergence result.

II. Preliminaries

Notations: For any $k \in \mathbb{N}$, $[k]$ is the set consisting of the first k natural numbers, that is, $[k] := \{1, \dots, k\}$. By $n < \infty$, we understand the number of autonomous agents. The communication scheme is represented by a weighted graph $G = ([n], E)$ where $[n]$ is the set of nodes and $E = \{w_{ij} : i, j \in [n]\}$ is the set of weighted edges. We denote the weighted degree of node i as $d_i = \sum_j w_{ij}$. The dynamics evolve in $\mathbb{R}^r$ for some $r \geq 1$ that is endowed with the inner product $\langle \cdot, \cdot \rangle$ and the Euclidean norm $\| \cdot \|$. Each agent $i \in [n]$ is characterized by the state $(x_i, v_i) \in \mathbb{R}^r \times \mathbb{R}^r$, where $x_i = (x_i^{(1)}, \dots, x_i^{(r)})$ and $v_i = (v_i^{(1)}, \dots, v_i^{(r)})$ stand for the position and velocity of i, respectively. Compact representation of the overall network state include $x = (x_1, \dots, x_n)$ and $v = (v_1, \dots, v_n)$ for elements x and v in the augmented space $\mathbb{R}^{nr}$. For $l \in [r]$, the spread of $y \in \mathbb{R}^{nr}$ in the lth dimension is

$$S_l(y) = \max_i y_i^{(l)} - \min_i y_i^{(l)} = \max_{i,j} |y_i^{(l)} - y_j^{(l)}|$$

and finally we denote $S(y) = \max_l S_l(y)$. Throughout this paper, we reserve the notation "$\cdot$" for the classic derivative and $\frac{d}{dt}$ for the right-hand Dini derivative.

The Contraction Coefficient: The problem of stability in asymmetric consensus networks is associated with the asymptotic behavior of products of non-negative matrices [24]. This behavior is, in turn, investigated with the use of the contraction coefficient. This instrumental notion to the theory of non-negative matrices, estimates the averaging effect of stochastic matrices when they act on vectors. Below, we present the following generalization: For a $P = [p_{ij}]$, non-negative $n \times n$ matrix with constant row sums (i.e., $\sum_j p_{ij} \equiv m$), it holds that

$$S(Pz) \leq \tau(P) S(z), \ \forall \, z \in \mathbb{R}^n$$

where

$$\tau(P) = m - \min_{i \neq i' \in [n]} \sum_{k=1}^{n} \min\{p_{ik}, p_{i'k}\} \tag{4}$$

is the contraction coefficient. A proof of this result can be found in [9]. While the framework is primarily compatible with discrete time dynamics, it can be adapted to continuous time systems, [23], [27], [28]. The contributions of this paper rely on auxiliary results (see Lemmas 5 and 11, below) which, in turn, occur as elaborate modifications of the contraction estimate (4).

III. Networks of Cucker–Smale Type With Deterministic Perturbations

Let n be a number of agents and $i \in [n]$ with the state $(x_i, v_i) \in \mathbb{R}^r \times \mathbb{R}^r$. The models we consider in this paper can be cast as the following system of equations:

$$\begin{aligned} \dot{x}_i^{(l)} &= v_i^{(l)} \\ \dot{v}_i^{(l)} &= \sum_{j=1}^{n} \big(w_{ij}(t, x) + b_{ij}^{(l)}(t, x, v)\big)\big(v_j^{(l)} - v_i^{(l)}\big), \ t \geq t_0 \end{aligned} \tag{5}$$

for $l \in [r]$, w_{ij} are the coupling terms and $b_{ij} : \mathbb{R} \times \mathbb{R}^l \times \mathbb{R}^l \to \mathbb{R}^l$ a nontrivial deterministic perturbation between agents j and i. This perturbation may, in general, depend on both time and state. We look at two meaningful scenarios of this general model, assuming corresponding special forms of b_{ij}. One form leads to flocking to a synchronization solution while the other form leads to collision-less velocity coordination.

A. Heterogeneous Synchronization Using Diffusion Processes

The first type of perturbation considers a situation where agents, in addition to the consensus averaging, attain an individual way of flying that is dictated by an internal dynamical behavior. For this case we take (5) with

$$b_{ij}^{(l)}(t, x, v) = \frac{g^{(l)}(t, v_i)}{v_j^{(l)} - v_i^{(l)}}.$$

Then, for $t_0 \in \mathbb{R}$, we are led to the initial value problem

$$\begin{aligned} \dot{x}_i &= v_i \\ \dot{v}_i &= g(t, v_i) + \sum_{j=1}^{n} w_{ij}(t, x)(v_j - v_i), t \geq t_0 \\ x_i(t_0) &= x_i^0, \ v_i(t_0) = v_i^0 \in \mathbb{R}^r, \text{ given.} \end{aligned} \tag{6}$$

An agent's velocity is affected by both the state of the other nodes and by an inherent dynamical process. The state of the other nodes affects the agent at a rate that depends on time and on the position vector x. It is, therefore, far from clear that the condition of Theorem 1 ensures convergence. Our objective is to reveal the interplay between the coupling forces of the consensus network, the initial configuration, and the potential instability induced by the internal dynamics through a new stability condition. We proceed by stating assumptions on the acceptable behavior of the internal dynamic system and conclude with a condition on the coupling functions.

Let for $t \geq t_0$ the initial value problem

$$\dot{z} = g(t, z), \ z(t_0) = z^0 \in \mathbb{R}^r \tag{7}$$

together with its solution $z = z(t, t_0, z^0)$ defined in a maximal interval $[t_0, T)$. The hypothesis below aims to establish the well posedness of z.

Assumption 2: The function $g(t, z)$ is defined in $\mathbb{R} \times V$, where V an open subset of $\mathbb{R}^r$. It is continuous in $t \in \mathbb{R}$ and attains continuous first derivative in $z \in V$.

We impose the following smoothness and boundedness conditions on the coupling weights.

Assumption 3: The following statements hold for the functions $w_{ij}(t,x)$:

1) They are continuous functions of t and continuously differentiable functions of x.
2) They satisfy

$$\infty > \overline{w} \geq \sup_{t\geq t_0} \sup_{x} \max_{i\neq j} w_{ij}(t,x) \geq \inf_{t\geq t_0} w_{ij}(t,x) \geq \psi(S(x))$$

for $\psi \geq 0$ an integrable and nonincreasing function.

The rates w_{ij} are uniformly bounded from above, but not from below as $\psi(\cdot)$ is allowed to vanish. This means that Assumption 3 allows the spatially decaying property. In addition, it allows for either symmetric or asymmetric couplings. For the statement of the first result, we are in need of some additional notation. Let

$$K(t,l,y,w) = \int_0^1 \frac{\partial}{\partial z^{(l)}} g^{(l)}(t, qy + (1-q)w)\, dq + \sum_{h\neq l} \left| \int_0^1 \frac{\partial}{\partial z^{(h)}} g^{(l)}(t, qy+(1-q)w)\, dq \right|$$

and for the maximal solution $\big(x(t), v(t)\big),\ t\in[t_0,T)$ of (6)

$$K = \sup_{t\in[t_0,T)} \max_{i,i'\in[n], l\in[r]} K(t,l,v_i(t),v_{i'}(t)). \tag{8}$$

This quantity represents the effect of the internal dynamic rule g in the coupling process. Evaluated on the maximal solution, K, is the worst case estimate that we must take into account in order to derive initial conditions that compensate for the potential instability that g will induce in the system that will in turn weaken the coupling rate w_{ij}. Observe also in $K(t,l,y,w)$ that the cross terms $\frac{\partial}{\partial z^{(h)}} g^{(l)}$ for $h \neq l$, are added in absolute value, as opposed to $\frac{\partial}{\partial z^{(l)}} g^{(l)}$. This discrepancy is the result of the dimensionality problem. Seeking synchronization of v_i in all r dimensions, one must take into account the effect that the internal dynamics g have in all r dimensions. Due to lack of structure on g, we have no choice but to regard the rate at which g varies in different dimensions ($h \neq l$) as a purely negative perturbation against the synchronization along the lth dimension.

Theorem 4: Consider the initial value problem (6) with Assumptions 2 and 3 to hold and its maximal solution $(x(t), v(t)),\ t\in[t_0,T)$. Assume also that
(1) there exists $d>0$ such that

$$S(v^0) < \int_{S(x^0)}^{d} \big(n\psi(r) - K\big)\, dr$$

(2) there exist $\varepsilon > 0$ such that

$$n\psi(d^*) \geq K + \varepsilon$$

for $d^* > 0 : S(v^0) = \int_{S(x^0)}^{d^*} \big(n\psi(r)-K\big)\, dr$ and K as in (8). Then, (x,v) satisfies

$$S(v(t)) \leq e^{-\varepsilon t} S(v^0) \ \ \&\ \ \sup_{t\geq t_0} S(x(t)) < d^*,\ \forall\, t\in[t_0,T).$$

The proof of the Theorem relies on the following result, the proof of which is put in the Appendix:

Lemma 5: Let (x,v) be the maximal solution of (6), defined in $[t_0,T)$. Then

$$\frac{d}{dt} S\big(v(t)\big) \leq \big[K - n\psi\big(S(x(t))\big)\big] S\big(v(t)\big) \tag{9}$$

for $t\in[t_0,T)$.

Proof of Theorem 4: Consider the functional

$$\mathcal{V}(x,v) = S(v) + \int_0^{S(x)} \big(n\psi(r) - K\big)\, dr \tag{10}$$

and evaluate it at the solution $\big(x(t),v(t)\big), t\in[t_0,T)$ with $\mathcal{V}(t) = \mathcal{V}\big(x(t),v(t)\big)$. From (1.) there exists $t_1 > t_0$ such that for $t \in [t_0,t_1)$

$$\frac{d}{dt}\mathcal{V}(t) \leq \frac{d}{dt} S(v(t)) + \big[n\psi\big(S(x(t)) - K\big] S\big(v(t)\big) \leq 0$$

in view of Lemma 5 and

$$\frac{d}{dt} S(x(t)) \leq S(v(t)).$$

The last claim is justified as follows: Note that $S(x(t)) = |x_i^{(l)}(t) - x_j^{(l)}(t)|$, for some $i,j\in[n]$ and $l\in[r]$ (possibly dependent on t). Then

$$\begin{aligned}\frac{d}{dt} S(x(t)) &= \frac{d}{dt}|x_i^{(l)}(t) - x_j^{(l)}(t)| \leq \left|\frac{d}{dt}\big(x_i^{(l)}(t) - x_j^{(l)}(t)\big)\right| \\ &= |v_i^{(l)}(t) - v_j^{(l)}(t)| \\ &\leq S(v(t))).\end{aligned}$$

Consequently, $\mathcal{V}(t) \leq \mathcal{V}(t_0)$ for $t < t_1$, equivalent to

$$S(v(t)) + \int_0^{S(x(t))} (n\psi(r) - K)\, dr \leq S(v^0) + \int_0^{S(x^0)} \big(n\psi(r) - K\big)\, dr$$

and obviously

$$\int_0^{S(x(t))} (n\psi(r) - K)\, dr \leq S(v^0) + \int_0^{S(x^0)} (n\psi(r) - K)\, dr.$$

Condition (1.) also implies the existence of $d^* < d$ as in condition (2.) the inequality above yields

$$\int_0^{S(x(t))} (n\psi(r) - K)\, dr \leq \int_0^{d^*} (n\psi(r) - K)\, dr$$

so

$$\int_{S(x(t))}^{d^*} (n\psi(r) - K)\, dr \geq 0.$$

The last inequality implies that $S(x(t)) \leq d^*$ for $t < t_1$ and $n\psi(d^*) - K \geq \varepsilon > 0$. Since no assumption was taken on t_1, the monotonicity of ψ yields that we can take $t_1 = T$ proving the second claim of the theorem. The differential inequality (9) then yields the first claim, concluding the proof. ■

This result establishes the connection between the internal and the position-dependent coupling dynamics. The power of Theorem 4 can be further extracted if we assume that the solution

v is *a priori* trapped within a region that possibly depends on the initial conditions. In such case, $T = \infty$ and the imposed conditions can be checked more easily. The following result asserts that for a particular type of compact subsets $\mathbb{R}^r$ into which z of (7) remains trapped, implies that v in (6) behaves likewise.

Theorem 6: Assume that U is a compact, convex g-invariant subset of $\mathbb{R}^r$. Then $v_i^0 \in U$ for $i \in [n]$, guarantees that the solution (x, v) of (6) exists for all times. In addition, the results of Theorem 4 hold true with K as in (8) substituted by

$$K = \sup_{t \geq t_0} \max_{y, w \in U, l \in [r]} K(t, l, y, w).$$

Proof: Consider the maximal solution $(x(t), v(t))$, $t \in [t_0, T)$ of (6). It suffices to show that $v_i(t_0) = v_i^0 \in U$ implies $v_i(t) \in U$ for all $t \in [t_0, T)$ and $i \in [n]$. Then T can be extended to ∞, establishing the solution (x, v) in the large. We can discretize the second part of (6) as follows:

$$v_i(t+k) = v_i(t) + k\left[g(t, v_i(t)) + \sum_{j=1}^{n} w_{ij}(t)\big(v_j(t) - v_i(t)\big)\right] \tag{11}$$

where $w_{ij}(t) = w_{ij}(t, x(t))$, $t \in [t_0, T)$, and $i \in [n]$. The result is then proved if we can show that $v_i(t+k) \in U$ for k small and arbitrary $t \in [t_0, T)$. The collection of the initial values $\{v_i^0\}$ lies in U, and the convexity of U implies that the convex hull of $\{v_1^0, \ldots, v_n^0\}$ is also a subset of U. Pick $i \in [n]$ and let's elaborate on $v_i(t+k)$ in (11), at $t = t_0$. By the g-invariance property of U, one can discretize the uncoupled equation and conclude that there is $k' > 0$ such that

$$v_i^a(t_0 + k) := v_i^0 + kg(t_0, v_i^0) \in U$$

for any $k \in [0, k')$. In addition, if on ignores the internal dynamics (i.e., $g \equiv 0$), Assumption 3 on the weights, implies that there is k'' such that

$$v_i^b(t_0 + k) := v_i^0 + k\sum_{j=1}^{n} w_{ij}(t_0)(v_j^0 - v_i^0) \in U$$

for $k \in [0, k'')$. The latter claim is because the right-hand side of the equation is a convex sum of $\{v_i^0\}$, thus places $v_i^b(t_0 + k)$ exactly in the convex hull of $\{v_1^0, \ldots, v_n^0\}$, which, in turn, is in U. We showed that both $v_i^a(t_0 + k)$ and $v_i^b(t_0 + k)$ are points in U for small enough $k \in [0, \min\{k', k''\})$. For U convex

$$U \ni \frac{1}{2}v_i^a(t_0 + k) + \frac{1}{2}v_i^b(t_0 + k) = v_i^0 + \frac{k}{2}\left[g(t_0, v_i^0) + \sum_{j=1}^{n} w_{ij}(t_0)(v_j^0 - v_i^0)\right].$$

Observe that the right-hand side is precisely this of (11) with $l = k/2$ and $t = t_0$. We remark that in view of Assumptions 2 and 3, k' and k'' can be chosen independent of t_0 or i so that the argument remains valid for any $t \in [t_0, T)$ and $i \in [n]$. ■

Remark 7: In Theorems 4 and 6, we assumed that every agent has access to the states of the rest of the agents. This is an all-to-all communication and it may be thought of as too demanding. The invoked argument allows for the following small relaxation: We can ask for every two agents that do not communicate, the existence of a third agent with which both of them must communicate. In this case, the estimates of Theorem 4 are valid if $n\psi(r)$ in the initial condition assumption is substituted by $\psi(r)$.

B. Flocking With Guaranteed Collision Avoidance

The second dynamic model we will study rolls back to the classic consensus problem and convergence to a common constant value. Here the alignment should be achieved with agents positioning themselves in at least a minimum distance from each other. For this, we will study a special case of (5) with

$$b_{ij}^{(l)}(t, x, v) = -\frac{1}{S(v)} f_{ij}(\|x_i - x_j\|^2)\langle x_i - x_j, v_i - v_j\rangle,\ l \in [r].$$

The protocol we propose is

$$\begin{aligned}
\dot{x}_i &= v_i \\
\dot{v}_i &= \sum_{j=1}^{n}\Bigg(w_{ij}(t, x) \\
&\quad - \frac{f_{ij}(\|x_i - x_j\|^2)\langle x_i - x_j, v_i - v_j\rangle}{S(v)}\Bigg)(v_j - v_i) \\
x_i(t_0) &= x_i^0,\ v_i(t_0) = v_i^0 \in \mathbb{R}^r,\ \text{given}
\end{aligned} \tag{12}$$

where w_{ij} as in Assumption 3 and the additional terms that model the collision prevention mechanism. The functions f_{ij} are repelling forces that can be appropriately constructed so as to keep the agents at a prescribed relative distance.

Assumption 8: For any $i \neq j$, $f_{ij}(\cdot)$ is a continuous nonnegative function defined in (d_0, ∞), for some constant $d_0 > 0$ such that

$$\int_{d_0}^{d_1} f_{ij}(r)\,dr = +\infty \text{ and } \int_{d_1}^{+\infty} f_{ij}(r)\,dr < +\infty$$

for all $d_1 > d_0$.

A simple example of symmetric repelling function (also to be used in §IV) is $f(r) = (r - d_0)^{-\varepsilon}$ for any fixed $\varepsilon > 1$. More examples can be found in [4]. The objective of this section is the derivation of sufficient conditions for flocking of (8). One should expect a formula that connects the coupling strength with the functions f_{ij}.

Theorem 9: Consider the initial value problem (12) with Assumptions 3 and 8 to hold, and its maximal solution $(x(t), v(t))$ for $t \in [t_0, T)$. If $i \neq j$ implies $\|x_i^0 - x_j^0\| > d_0$ and

$$\frac{S(v^0)}{n} < \frac{1}{2}\int_{S(x^0)}^{\infty} \psi(r)\,dr - \max_{i \neq j}\int_{\|x_i^0 - x_j^0\|^2}^{\infty} f_{ij}(r)\,dr, \tag{13}$$

then

1) $T = \infty$,
2) $\|x_i(t) - x_j(t)\| > d_0$, for all $t \geq t_0$,
3) the solution satisfies

$$S(v(t)) \to 0 \text{ as } t \to \infty \text{ and } \sup_{t \geq t_0} S(x(t)) < \infty.$$

The proof of Theorem 9 follows partly the steps of the proof of Theorem 4 and partly the arguments developed in [4].[1]

We begin with a preliminary result that establishes a solution estimate of v similar to Lemma 5. Both the preliminary and the main results rely on the following observation.

Remark 10: Note that $b_{ij}(t,x,v)$ can be written as

$$b_{ij}(t,x,v)=\frac{1}{2S(v)}\frac{d}{dt}\int_{\|x_i(t)-x_j(t)\|^2}^{\infty} f_{ij}(r)\,dr. \tag{14}$$

Given a solution (x,v) of (12) defined in $t\in[t_0,T)$ we set $h=h_t$ and $h^{'}=h_t^{'}\in[n]$ the agents that lie closes to each other, that is, the indices that minimize $||x_i-x_{i'}||$. While this mapping may not be unique, it is a piece-wise constant function of t.

Lemma 11: The maximal solution (x,v) of (12) defined in $[t_0,T)$ satisfies for all $i,i^{'}\in[n]$ and $l\in[r]$

$$\frac{d}{dt}|v_i^{(l)}-v_{i'}^{(l)}|\leq -m|v_i^{(l)}-v_{i'}^{(l)}|+(m-\rho_{i,i'})S(v)-\Gamma_{i,i'} \tag{15}$$

where $m=m(t)$ is an arbitrary but fixed, non-negative, integrable function, $\rho_{i,i'}=\sum_{j=1}^{n}\min\{w_{ij}(t,x(t)),w_{i'j}(t,x(t))\}$ and

$$\Gamma_{i,i'}=\frac{1}{2}\sum_{j\neq i}\min\left\{\frac{d}{dt}\int_{||x_i-x_j||^2}^{\infty} f_{ij}(r)\,dr,\right.$$
$$\left.\frac{d}{dt}\int_{||x_{i'}-x_j||^2}^{\infty} f_{i'j}(r)\,dr\right\}.$$

The proof of Lemma 11 is put in the Appendix. When i and $i^{'}$ are chosen so as to maximize $\max_l|v_i^{(l)}-v_{i'}^{(l)}|=S(v)$ then one obtains the estimate

$$\frac{d}{dt}S(v)\leq-\rho_{i,i'}S(v)-\Gamma_{i,i'}. \tag{16}$$

Proof of Theorem 9: For the reader's convenience, we make the proof to consist of the following four steps: Collision avoidance conditions for the maximal solution, existence of the maximal solution for all times, persistent connectivity of the flock, and asymptotic velocity alignment.

a) Collision Avoidance: Define $\mathcal{E}:\mathbb{R}^{nr}\times\mathbb{R}^{nr}\to\mathbb{R}$

$$\mathcal{E}(x,v)=S(v)+|v_h^{(l)}-v_{h'}^{(l)}|+\sum_{j=1}^{n}\gamma_j^{i,i'}+\sum_{j=1}^{n}\gamma_j^{h,h'}$$

where $l\in[r]$ arbitrary but fixed, $h,h^{'}\in[n]$ are as in Remark 10, $i,i^{'}\in[n]$ are chosen to maximize the velocity diameter, and consequently satisfy (16), and finally

$$\gamma_j^{i,i'}=\gamma_j^{i,i'}(x,v)=\int_{||x_k-x_j||^2}^{\infty} f_{kj}(r)\,dr \text{ with } k=k_{i,i'}:$$
$$f_{kj}(||x_k-x_j||^2)\langle x_k-x_j,v_k-v_j\rangle$$
$$=\min\left\{f_{ij}(||x_i-x_j||^2)\langle x_i-x_j,v_i-v_j\rangle,\right.$$
$$\left.f_{i'j}(||x_{i'}-x_j||^2)\langle x_{i'}-x_j,v_{i'}-v_j\rangle\right\}.$$

[1] In fact Theorem 9 may be considered as the asymmetric alternative of [4].

All of these indices are potentially time dependent but piecewise constant. Differentiating $\mathcal{E}$ along $(x(t),v(t)),t\geq t_0$, Lemma 11 for $m(t)=\rho_{i,i'}(t,x(t))$ and (16) will yield

$$\begin{aligned}\frac{d}{dt}\mathcal{E}(x(t),v(t))=&-\rho_{i,i'}S(v)-\Gamma_{i,i'}-\rho_{i,i'}|v_h^{(l)}-v_{h'}^{(l)}|\\&+\left(\rho_{i,i'}-\rho_{h,h'}\right)S(v)-\Gamma_{h,h'}\\&+\frac{d}{dt}\sum_{j=1}^{n}\gamma_j^{i,i'}(x(t),v(t))\\&+\frac{d}{dt}\sum_{j=1}^{n}\gamma_j^{h,h'}(x(t),v(t))\\=&-\rho_{i,i'}|v_h^{(l)}-v_{h'}^{(l)}|-\rho_{h,h'}S(v)\leq 0.\end{aligned}$$

Note that

$$\text{either}\int_{\|x_h(t)-x_{h'}(t)\|^2}^{\infty} f_{hh'}(r)\,dr \text{ or } \int_{\|x_h(t)-x_{h'}(t)\|^2}^{\infty} f_{h'h}(r)\,dr$$

is a member of the sum $\sum_j\gamma_j^{h,h'}$. Then, $\mathcal{E}(t)\leq\mathcal{E}(t_0)$ implies that for the two agents h and h' that are in closest distance from each other

$$\min\left\{\int_{\|x_h(t)-x_{h'}(t)\|^2}^{\infty} f_{hh'}(r)\,dr,\int_{\|x_h(t)-x_{h'}(t)\|^2}^{\infty} f_{hh'}(r)\,dr\right\}$$
$$\leq\mathcal{E}(t)\leq\mathcal{E}(t_0)<\infty.$$

Since both $f_{hh'}$ and $f_{h'h}$ satisfy Assumption 8, we proved that $\|x_i(t)-x_j(t)\|\geq d^*$ for some $d^*>d_0$ for all $i\neq j\in[n]$ and $t\in[t_0,T)$.

b) Existence in the Large: From $\mathcal{E}(t)\leq\mathcal{E}(t_0)$, we also deduce that $S(v(t))\leq\mathcal{E}(t_0)<\infty$ hence

$$S(x(t))\leq S(x^0)+T\mathcal{E}(t_0):=\bar{X}$$

and the solution lies for $[t_0,T)$ in

$$\Omega=\{(x,v):S(x)\leq\bar{X},\|x_{i,j}\|\geq d^*,i\neq j,S(v)\leq\mathcal{E}(t_0)\}$$

where d^* defined in the first part of the proof. However, Ω is a compact subset of

$$\{(x,v):S(x)\leq\bar{X},\|x_{i,j}\|>d_0,i\neq j\}.$$

The fundamentals in the theory of differential equations assure, however, that this cannot occur if $T<\infty$ (see for example [14, Th. 1.21]) hence, it is ensured that the solution is eventually defined for all $t\geq t_0$.

c) Bounded Flock: We recall from the first step that

$$\frac{d}{dt}\mathcal{E}(x(t),v(t))\leq-\rho_{i,i'}|v_h^{(l)}-v_{h'}^{(l)}|-\rho_{h,h'}S(v)\leq-\rho_{h,h'}S(v).$$

If we integrate from t_0 to t and use the lower bound of $\rho_{h,h'}$ taken in view of Assumption 3, we obtain

$$\mathcal{E}(t)-\mathcal{E}(t_0)\leq-\int_{t_0}^{t}n\psi\big(S(x(r))\big)S(v(r))\,dr\Rightarrow$$

$$\mathcal{E}(t_0)\geq\int_{t_0}^{t}n\psi\big(S(x(r))\big)S(v(r))\,dr. \tag{17}$$

If we set $p(s) = S(x(s))$, we observe that $\frac{dp}{ds} \leq S(v(s))$ and deduce

$$\int_{S(x^0)}^{S(x(t))} n\psi(s)\, ds \leq \mathcal{E}(t_0).$$

Had the flock been dissolved, there should be a sequence $\{t_i\}_{i\geq 0}$ with $t_i \to \infty$ as $i \to \infty$, so that $S(x(t_i)) \to \infty$ as $t_i \to \infty$. This would mean that

$$\int_{S(x^0)}^{\infty} n\psi(r)\, dr \leq \mathcal{E}(t_0)$$

$$\leq 2S(v^0) + 2n \max_{i\neq j} \int_{||x_i^0 - x_j^0||^2}^{\infty} f_{ij}(r)\, dr$$

that is not possible in view of (13). Thus

$$\sup_{t\geq t_0} S(x(t)) < \infty \tag{18}$$

that is, the flock remains connected.

d) Convergence to Flocking: At first, we combine (17) and (18) to conclude

$$\int^{\infty} S(v(r))\, dr < \infty.$$

Moreover, we used $\frac{d}{dt}\mathcal{E}(x(t), v(t)) < 0$ to show that $\sup_{t\geq t_0} S(v(t)) < \mathcal{E}(t_0)$ and this implies that $|S(v(t))|$ is uniformly bounded. It only remains to show that $S(v(t))$ is uniformly continuous so that Barbalat's Lemma ([12, Lemma 8.2]) applies to $k(t) := \int_{t_0}^{t} S(v(s))\, ds$ to conclude

$$\dot{k}(t) = S(v(t)) \to 0 \text{ as } t \to \infty.$$

The last condition is shown by direct application of the definition of uniform continuity: For any $t_1, t_2 \geq t_0$ close to each other, there exist $i, i' \in [n]$ and $l \in [r]$ such that

$$\begin{aligned}
&|S(v(t_1)) - S(v(t_2))| \\
&\quad = |(v_i^{(l)}(t_1) - v_{i'}^{(l)}(t_1)) - (v_i^{(l)}(t_2) - v_{i'}^{(l)}(t_2))| \\
&\quad \leq 2 \max_{h\in\{i,i'\}} |v_h^{(l)}(t_1) - v_h^{(l)}(t_2)| \\
&\quad \leq 2 \max_{h\in\{i,i'\}} |\dot{v}_h^{(l)}(t^*)| \cdot |t_1 - t_2|.
\end{aligned}$$

Since $\dot{v}_h^{(l)}(t)$ satisfies (12), its absolute value is bounded above by finite number of terms $|w_{ij}| \leq \overline{w}$, $|b_{ij}(x, v)| \leq \max_{ij} f_{ij}(\underline{d}^2)\big(\sup_t S(x(t))\big)^2 \mathcal{E}(t_0)$ where $\underline{d} = \inf_{t\geq t_0} \min_{i\neq j} |x_{ij}(t)| > d_0$ and $|S(v(t))| \leq \mathcal{E}(t_0)$, each of which is independent of time and the uniform continuity property holds true. Barbalat's lemma can then be applied concluding the proof. ■

IV. Examples and Simulations

In this section, we present applications of our rigorous results. We will use a group of $n = 5$ and examine networks of different dimensions and types of solutions. The initial time is set at $t_0 = 0$. The coupling rates are set $w_{ij}(t, x) = w\frac{1.5+0.5\sin(t)}{(|x|+\beta_{ij}^2)^\delta}$ for $\delta \geq 0$, $\beta_{ij} \in (0, \sqrt{2})$ and $w \geq 0$ a uniform control parameter.[2]

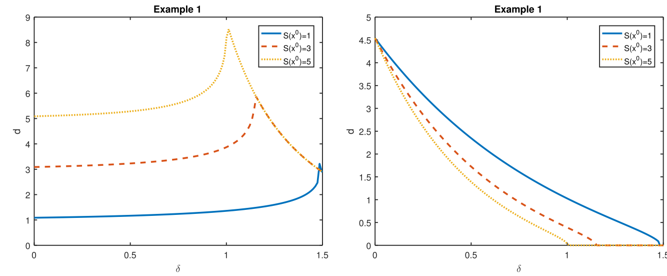

Fig. 1. Stability condition curves for different values of $S(x^0)$. The larger the initial spread of the relative positions, the weaker the couplings initially are.

A. Scalar Networks

We consider (7) with $g(t, z) = \cos(t)(z - 1)(z - 2)$. Its solution $z(t) = z(t, 0, z^0),\ t \geq 0$ is

$$z(t) = \frac{2 - \frac{z_0 - 2}{z_0 - 1} e^{\sin(t)}}{1 - \frac{z_0 - 2}{z_0 - 1} e^{\sin(t)}}.$$

It can be easily checked that for $z^0 \in (1, 2)$ the solution exists for all times and is periodic with period 1. We consider now the network (6) and its maximal solution (x, v) with initial setup $v^0 = (1.2, 1.4, 1.1, 1.5, 1.3)$. We select $w = 1$ throughout this example. The solution (x, v) exists for all times remaining trapped in $U = [1, 2]$ but we can in fact say much more. Due to the monotonicity of solution z, using the same arguments as in the proof of Theorem 6, we can calculate the estimate K by evaluating it over $z(t, 0, 1.5)$, since no solution v_i will essentially exceed $z(t, 0, 1.5)$. Thus we estimate K as

$$K \leq 2\frac{2 + e^{\sin(t)}}{1 + e^{\sin(t)}} - 3 \leq \frac{1 - e^{-1}}{1 + e^{-1}} \approx 0.462.$$

In addition, $w_{ij}(t, x) \geq \frac{1}{(|x|+2)^\delta}$. From the conditions of Theorem 6, exponential convergence to flocking with rate $\varepsilon > 0$ can occur if we can find number d such that

$$0.4 < 5\frac{(d+2)^{(1-\delta)} - (S(x^0)+2)^{(1-\delta)}}{1-\delta} - 0.462(d - S(x_0))$$

and

$$\frac{5}{(d^*+2)^\delta} - 0.462 > 0$$

for d^* that after substituting it to d, it can achieve equality in the first condition. At $\delta = 0$, one can easily verify that the first condition holds true for every d^* and that the first inequality is always satisfied for d large enough.

As a brief numerical exploration we took $S(x^0) = 1, 3, 5$ and examined the two conditions one after the other. The results are plotted in Fig. 1 and suggest that all the conditions can be satisfied for $\delta < 1.1$. Fig. 2 presents realizations of (x, v) with

[2]All simulations are carried with the `ode23` routine in MATLAB.

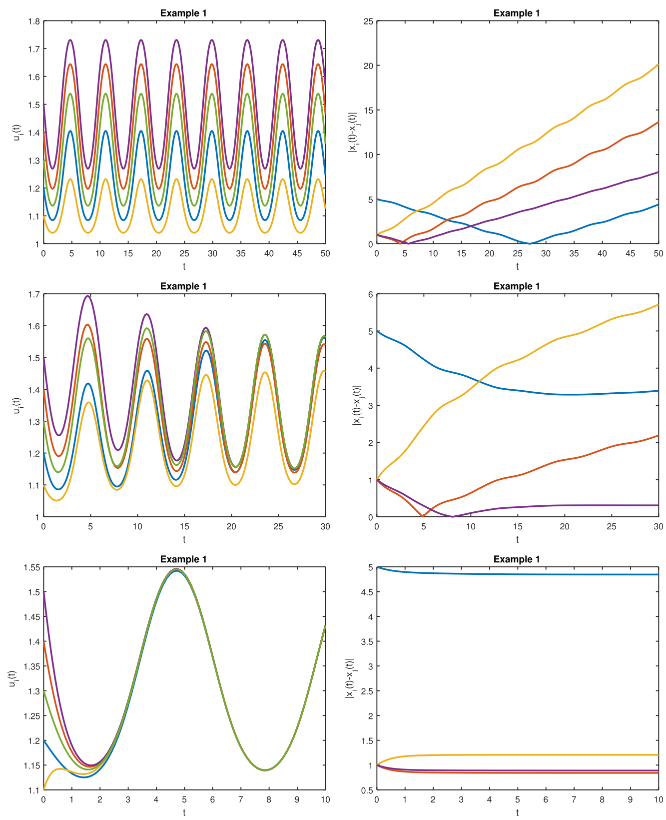

Fig. 2. Realization of (x, v) of Example 1 with $\delta = 10$, $\delta = 4$, and $\delta = 0.9$. The first column presents v_i, $i \in [5]$ and the second column presents the differences $|x_i - x_j|$, $i < j$.

$S(x^0) = 5$. The first simulation depicts a network with the weak coupling $\delta = 10$, where no synchronization can happen. In the second attempt, we adopt a stronger coupling with $\delta = 4$ that, however, still does not yield a feasible d to satisfy our conditions. Finally for $\delta = 0.9$, we can see synchronization of solutions that occur exponentially fast.

B. Example 2. Chaotic Flocking

The next example is on the problem of synchronizing chaotic oscillators. The nominal equation is chosen to be the Lorenz system

$$g(z^{(1)}, z^{(2)}, z^{(3)}) = \begin{bmatrix} 10(z^{(2)} - z^{(1)}) \\ -z^{(2)} + z^{(1)}(28 - z^{(3)}) \\ -\dfrac{8}{3} z^{(3)} + z^{(1)} z^{(2)} \end{bmatrix}.$$

The solutions of (7) converge for almost all initial values to a strange attractor [30]. This means that there is a g-invariant, convex subset $U \subset \mathbb{R}^3$ that includes the limit set. It can be verified that $U \subset \tilde{U} = [-17, 17.5] \times [-22, 24.5] \times [7, 45]$. Furthermore, $\tilde{U}$ can be numerically verified to be g-invariant while it is clearly compact and convex. We can apply Theorem 6 to (6) with g as in this example and initial conditions in $v_i^0 \in \tilde{U}$, $i \in [5]$. Then, we can calculate an estimate on K

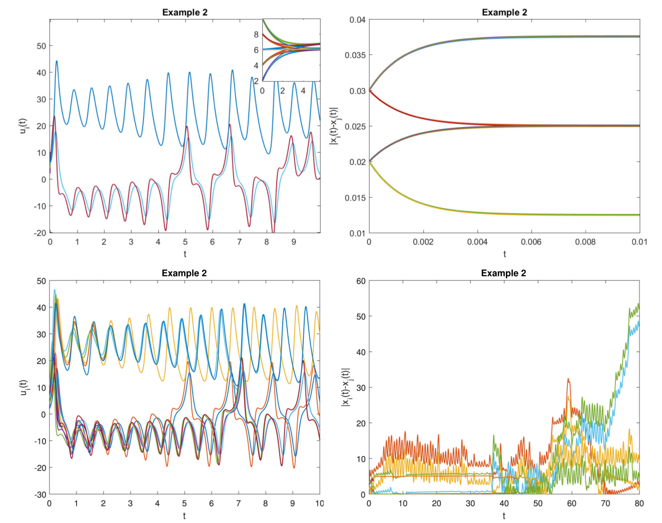

Fig. 3. Simulation Example 2. Synchronization of chaotic oscillators with $r = 3$. In the first two figures, we observe extremely strong synchronization under the theoretical sufficient conditions, $w = 150$ and $\delta = 0.5$. In the two figures that follow, we present loose coupling with the same w and $\delta = 7$. We observe the failure of our network to synchronize.

based[3] on U

$$K \leq \max_{z \in \tilde{U}} \left\{ 0, |28 - z^{(3)}| - 1 + |z^{(1)}|, |z^{(2)}| + |z^{(1)}| - \frac{8}{3} \right\} \approx 39.4$$

We set $S(v^0) = S(x^0) = 9$ and Theorem 6 applies if one can find d such that

$$9 < \int_9^d \left(\frac{5w}{(r+2)^\delta} - 39.4 \right) dr \text{ and } \frac{5w}{(d^* + 2)^\delta} - 39.4 > 0$$

where d^* when substitutes d in the first condition achieves equality. Note that for any δ small enough one can always find d^* such that

$$9 + 39.4(d^* - 9) > 39.4(d^* + 2)^\delta \int_9^{d^*} \frac{dr}{(r+2)^\delta}.$$

This is a relation after solving for w in the first condition (that is an equality for $d = d^*$) and substituting in the second condition. If, for example we take $\delta = 0.5$, then the last condition is satisfied with $d^* = 11.67$ and $w \approx 150$. Fig. 3 depicts our results for strong $\delta = 0.5$ and loose $\delta = 7$ coupling, respectively. The rest of the parameters w and initial configurations were kept identical. These two illustrative choices represent the case that conditions of Theorem 6 are met and when the conditions are violated, respectively.

C. Example 3. Collision Avoidance

We conclude this section with an illustration of Theorem 9. We consider the network (12) with $r = 2$. The repelling func-

[3]Clearly, better estimates on K may be achieved when $\tilde{U}$ is substituted with a sharper estimate of U.

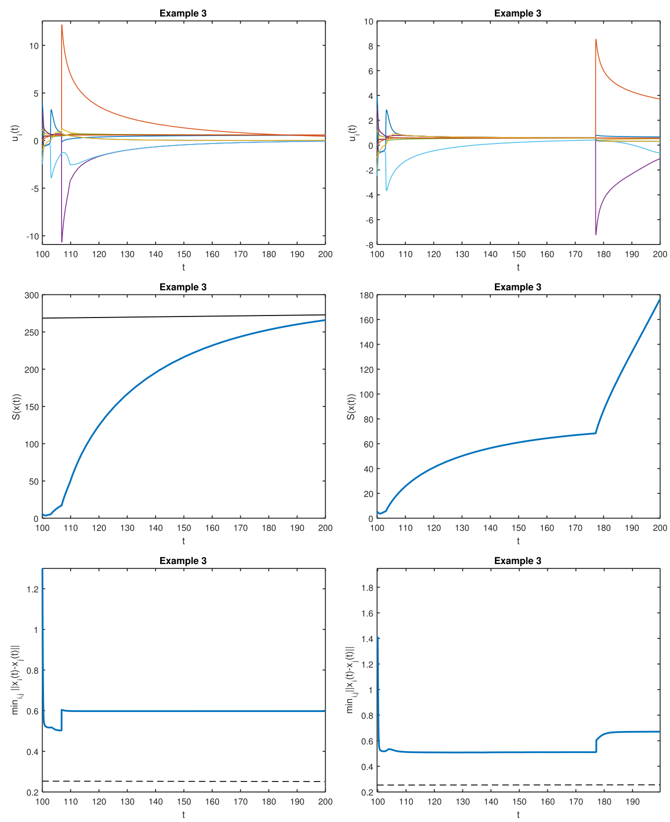

Fig. 4. Simulation Example 3. Flocking with collision avoidance. The first column is a strong coupling simulation of the velocity curves, the maximum distance, and the minimum distance. The second column is a weak coupling simulation with the same quantities. While both simulations achieve collision avoidance, only the first one achieves flocking.

tions are taken

$$f_{ij}(r) = \frac{C_{ij}}{(r - r_0)^{\varphi}}$$

for numbers C_{ij} arbitrarily chosen from $(1,2)$, $\varphi = 1.5$, and $r_0 = 0.25$. Our initial data are $S(v^0) = 6$ and $S(x^0) = 4$. The first simulation runs with $\delta = 1$ so that the condition of Theorem 9 is clearly satisfied and asymptotic flocking with collision avoidance is achieved. See Fig. 4 for the simulation results. The repelling functions destabilize the agents' velocities in an uncontrollable manner. It then takes the strength of the coupling networks to determine the stability of flocking solutions. In the first case, $\delta = 1$ ensures the stability because it makes the right-hand side of condition of Theorem 9 unbounded.

V. Discussion

We addressed two variations of the classic nonlinear flocking algorithm of Cucker–Smale type with asymmetric coupling rates. The striking similarity in the analysis of these two evidently different algorithms is clear: Both networks include the consensus-based stabilizing term and a potentially destabilizing term. The perturbations affect the stability of the network because it leads to fluctuations in the relative velocities $v_i - v_j$. This, in turn, results in weaker coupling strength, the level of which can dissolve the network to fail to converge to consensus. This interplay is reflected in the initial configuration sufficient conditions that ensure convergence of the overall network to flocking behavior.

Nonlinearity imposes investigation of solutions that may not exist for all times. Theorem 4 accepts in principal such solutions. Since this event is of little interest to stability problems, Theorem 9 concerns nominal systems with solutions that exist for all times.

We focused on reasonable forms of b_{ij} that correspond to real-world mathematical models. It is yet to be investigated what type of dynamics emerge with abstract nonlinear perturbations. One could reasonably assume that positive valued b_{ij}'s in (5) tend to stabilize the network toward a common constant equilibrium, while negative valued perturbation destabilize the network possibly toward more complex behavior.

A. Further Nonlinearities

Models (6) and (12) can be easily extended to nonlinearities in the spirit of (1.2) or (1.3). In this case, one would require additional assumptions on the type of these nonlinearities. For instance, note that (1.3) can be rewritten as

$$\dot{z}_i = \sum_{j=1}^{n} w_{ij}(t,z)(z_i - z_j)$$

where $w_{ij}(t,z) = \int_0^1 \frac{\partial}{\partial z} w_{ij}(t, q z_i - (1-q) z_j)\, dq$. Then, typical smoothness or uniformity assumptions on the nonlinear w_{ij} can make a second order version possible to be analyzed with our framework, under appropriate modifications. Similar arguments can be adopted for couplings like (1.2).

B. Problem of Dimensionality

For dynamics in one dimension, all metrics boil down to the absolute value making the analysis simple and elegant. In higher dimensional systems, one must carefully choose the most appropriate metric that suits the geometry of the generated solutions. Our analysis dictates the option of $S(y)$ and any discrepancy with the solutions of the nominal dynamics may result in more conservative convergence conditions (i.e., larger K). Attempts toward sharper sufficient conditions should probably allow some structure on the nominal dynamics g. In this case, the researcher can leverage on the geometry of the nominal solutions and their qualitative properties so as to come up with an appropriate contraction metric for the synchronization analysis.

C. Connection Topologies

A careful inspection of (4) reveals that the contraction coefficient that provides nontrivial estimates of the stochastic matrix P, must be associated with a graph that is rich in edges. In particular, it is asked that for every couple of two agents, there is at least one agent that affects both of the aforementioned two. The connectivity we assumed in this paper corresponds to a complete graph and clearly covers this case. The interest in networks of Cucker–Smale type lies primarily on the strength of the coupling weights and not on the topological structure. That is why they are usually assumed under all-to-all communication schemes. Looser connectivities are feasible for the synchronization network (6) but it involves more elaborate arguments, perhaps along the lines of [27]. This problem remains of interest to us and it will be considered in the future.

Acknowledgement

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Appendix

Proof of Lemma 5: Let $(x(t), v(t)),\ t \in [t_0, T)$ the maximal solution of (6). Fix $i, i' \in [n],\ l \in [r]$ and take $m = \bar{n}\overline{w} + K$, where $\bar{n}$ any number greater than n and $\overline{w}$ as in Assumption 3. From the second line of (6), we have

$$\dot{v}_j^{(l)} = -mv_j^{(l)} + \big(m - d_j\big)v_j^{(l)} + \sum_{k=1}^{n} w_{jk}v_j^{(l)} + g^{(l)}(t, v_j)$$

for $j = i, i'$, $w_{jk} = w_{jk}\big(t, x(t)\big)$, $d_j = \sum_{k\neq j} w_{jk}$. Set $\Delta_{i,i'}(t) = e^{-m(t-t_0)} \frac{d}{dt}\big(e^{m(t-t_0)}[v_i^{(l)}(t) - v_{i'}^{(l)}(t)]\big)$ and substituting the equations on i and i' we obtain

$$\Delta_{i,i'}(t) = \sum_{k=1}^{n}(\tilde{w}_{ik} - \tilde{w}_{i'k})v_k^{(l)} +$$
$$+ \sum_{h\neq l}\left[\int_0^1 \frac{\partial}{\partial z^{(h)}} g^{(l)}(t, qv_i + (1-q)v_{i'})\, dq\right]\big(v_i^{(h)} - v_{i'}^{(h)}\big)$$

where $\tilde{w}_{ij} = \tilde{w}_{ij}(t, l)$ defined as

$$\tilde{w}_{ij} = \begin{cases} m - d_i + c_{i,i'}, & j = i \\ w_{ij}, & j \neq i. \end{cases}$$

for $c_{i,i'} = \int_0^1 \frac{\partial}{\partial z^{(l)}} g^{(l)}(t, qv_i(t) + (1-q)v_{i'}(t))\, dq$. Similarly for $\tilde{w}_{i'j}$. Take $a_j = (\tilde{w}_{ij} - \tilde{w}_{i'j})$ and observe that

$$\sum_{j=1}^{n} a_j = \sum_{j\neq i} w_{ij} + w_{ii} - \sum_{j\neq i'} w_{i'j} - w_{i'i'}$$
$$= d_i + m - d_i + c_{i,i'} - d_{i'} - m + d_{i'} - c_{i,i'} = 0.$$

The index j for which $a_j > 0$ is denoted by j^+ and the index for which $a_j \leq 0$ is denoted by j^-. Set $\theta = \theta(t)$

$$\theta = \sum_{j^+} a_{j^+} = \sum_{j^+} |a_{j^+}| = -\sum_{j^-} a_{j^-} = \sum_{j^-} |a_{j^-}|$$
$$= \frac{1}{2}\sum_{j=1}^{n} |a_j| = \frac{1}{2}\sum_{j=1}^{n} |\tilde{w}_{ij} - \tilde{w}_{i'j}|$$

Then for $t \in [t_0, T)$

$$\Delta_{i,i'}(t) = \theta\left(\frac{\sum_{j^+} |a_{j^+}| v_{j^+}}{\theta} - \frac{\sum_{j^-} |a_{j^-}| v_{j^-}}{\theta}\right)$$
$$+ \sum_{h\neq l}\left[\int_0^1 \frac{\partial}{\partial z^{(h)}} g^{(l)}(t, qv_i + (1-q)v_{i'})\, dq\right]\big(v_i^{(h)} - v_{i'}^{(h)}\big)$$
$$\leq \left(\theta + \sum_{h\neq l}\left|\int_0^1 \frac{\partial}{\partial z^{(h)}} g^{(l)}(t, qv_i + (1-q)v_{i'})\, dq\right|\right) S(v)$$

But from the identity $|x - y| = x + y - 2\min\{x, y\}$ and the particular m we chose, it can be deduced that

$$\theta = m + c_{i,i'} - \sum_{k\neq i,i'} \min\{w_{ik}(t), w_{i'k}(t)\}.$$

So that

$$\Delta_{i,i'}(t) \leq \left(m + K - \sum_{k\neq i,i'} \min\{w_{ik}(t), w_{i'k}(t)\}\right) S(v).$$

Finally, for i, i' and l that maximize $|v_i^{(l)}(t) - v_{i'}^{(l)}(t)|$, we have

$$\begin{aligned} \frac{d}{dt} S(v(t)) &= \frac{d}{dt}\left[e^{-m(t-t_0)} S\big(e^{m(t-t_0)} v(t)\big)\right] \\ &= -mS(v(t)) + \Delta_{i,i'}(t) \\ &\leq \big[K - n\psi\big(S(x(t))\big)\big] S\big(v(t)\big). \end{aligned}$$

that is true for $t \in [t_0, T)$. ■

Proof of Lemma 11: Consider the maximal solution $\big(x(t), v(t)\big)$ on $[t_0, T)$. In any $l \in [r]$ dimension, $v_i^{(l)}$ satisfies

$$\dot{v}_i = -mv_i + (m - d_i)v_i + \sum_{j=1}^{n}(w_{ij} + b_{ij})v_j$$

where $w_{ij} = w_{ij}(t, x(t))$, $b_{ij} = b_{ij}(t, x(t), v(t))$ as in (14), $d_i = d_i(t) = \sum_j (w_{ij} + b_{ij})$ and $m = m(t)$ to be determined below. Note that we removed the l dependency because the dynamics are identical in all $[r]$ dimensions. We set $\phi(t, t_0) = \exp\{\int_{t_0}^{t} m(s)\ \ ds\}$. Following the proof of Theorem 4, $\Delta_{i,i'}(t) := \phi^{-1}(t, t_0)\frac{d}{dt}\big[\phi(t, t_0)|v_i(t) - v_{i'}(t))|\big]$, yields

$$\Delta_{i,i'}(t) = \sum_j \alpha_j v_j$$

for

$$\alpha_j = \begin{cases} (w_{ij} - w_{i'j}) + (b_{ij} - b_{i'j}), & j \neq i, i' \\ m - d_i - w_{ii'} - b_{i'i}, & j = i, \\ -m + d_{i'} + w_{i'i} + b_{ii'}, & j = i'. \end{cases}$$

Let j' denote the indices j for which $\alpha_j > 0$ and j'' the indices for which $\alpha_j \leq 0$. Then

$$\Delta_{i,i'}(t) = \sum_{j'} |\alpha_{j'}| v_{j'} - \sum_{j''} |\alpha_{j''}| v_{j''}.$$

Again, it holds that $\sum_j \alpha_j \equiv 0$, so we take $\theta_{ii'} = \theta_{ii'}(t)$ to be the sum of the positive α_j

$$\begin{aligned} \theta_{ii'} &= \sum_{j'} |\alpha_{j'}| = -\sum_{j''} \alpha_{j''} = \sum_{j''} |\alpha_{j''}| = \frac{1}{2}\sum_j |\alpha_j| \\ &\leq \frac{1}{2}\sum_{j=1}^{n} |w_{ij} - w_{i'j}| + \frac{1}{2}\sum_{j=1}^{n} |b_{ij} - b_{i'j}| \\ &= m - \sum_{j\neq i} \min\{w_{ij}, w_{i'j}\} - \sum_{j\neq i} \min\{b_{ij}, b_{i'j}\} \end{aligned}$$

in view of the identity $2\min\{x, y\} = x + y - |x - y|$.

$$\Delta_{i,i'}(t) = \theta\left(\frac{\sum_{j'} |w_{j'}| v_{j'}}{\sum_{j'} |w_{j'}|} - \frac{\sum_{j''} |w_{j''}| v_{j''}}{\sum_{j''} |w_{j''}|}\right).$$

Invoking the estimate on θ

$$\Delta_{i,i'}(t) \leq \big(m - \sum_{j\neq i} \min\{w_{ij}, w_{i'j}\} - \sum_{j\neq i} \min\{b_{ij}, b_{i'j}\}\big) S(v)$$

$$= \big(m - \rho_{i,i'}\big) S(v) - \Gamma_{i,i'}$$

where $\rho_{i,i'}$, $\Gamma_{i,i'}$ as in the statement of the Lemma. Observe that $\frac{d}{dt}|v_i(t) - v_{i'}(t)| = -m|v_i - v_{i'}(t)| + \Delta_{i,i'}(t)$ and substitute the estimate of $\Delta_{i,i'}(t)$ to conclude. ■

References

[1] M. Arcak, "Passivity as a design tool for group coordination," *IEEE Trans. Autom. Control*, vol. 52, no. 8, pp. 1380–1390, Aug. 2007.

[2] M. Ballerini *et al.*, "Interaction ruling animal collective behavior depends on topological rather than metric distance," *Proc. Nat. Acad. Sci. USA*, vol. 105, pp. 1232–1237, 2008.

[3] F. R K Chung, *Spectral Graph Theory*, vol. 92. Providence, RI, USA: Amer. Math. Soc., 1997.

[4] F. Cucker and J. G. Dong, "A general collision-avoiding flocking framework," *IEEE Trans. Autom. Control*, vol. 56, no. 5, pp. 1124–1129, May 2011.

[5] F. Cucker and S. Smale, "Emergent behavior in flocks," *IEEE Trans. Autom. Control*, vol. 52, no. 5, pp. 852–862, May 2007.

[6] F. Cucker and S. Smale, "On the mathematics of emergence," *Jpn. J. Math.*, vol. 2, pp. 197–227, 2007.

[7] J. M. González-Miranda, *Synchronization and Control of Chaos: An Introduction for Scientists and Engineers*. London, U.K: Imperial College Press, 2004.

[8] S-Y. Ha and J. G. Liu, "A simple proof of the cucker-smale flocking dyanics and mean-field limit," *Commun. Math. Sci.*, vol. 7, no. 2, pp. 297–325, 2009.

[9] D. J. Hartfiel, *Markov Set-Chains*, vol. 1695. New York, NY, USA: Springer, 1998.

[10] J. M. Hendrickx and J. N. Tsitsiklis, "Convergence of type-symmetric and cut-balanced consensus seeking systems," *IEEE Trans. Autom. Control*, vol. 58, no. 1, pp. 214–218, Jan. 2013.

[11] A. Jadbabaie, J. Lin, and A. S. Morse, "Coordination of groups of mobile autonomous agents using nearest neighbor rules," *IEEE Trans. Autom. Control*, vol. 48, no. 6, pp. 988–1001, Jun. 2003.

[12] H. K. Khalil, *Nonlinear Systems*. Englewood Cliffs, NJ, USA: Prentice-Hall, 2000.

[13] S. Manfredi and D. Angeli, "Necessary and sufficient conditions for consensus in nonlinear monotone networks with unilateral interactions," *Automatica*, vol. 77, pp. 51–60, 2017.

[14] N. G. Markley, *Principles of Differential Equations*. Hoboken, NJ, USA: Wiley, 2004.

[15] S. Martin, "Multi-agent flocking under topological interactions," *Syst. Control Lett.*, vol. 69, pp. 53–61, 2014.

[16] S. Martin, A. Girard, A. Fazeli, and A. Jadbabaie, "Multiagent flocking under general communication rule," *IEEE Trans. Control Netw. Syst.*, vol. 1, no. 2, pp. 155–166, Jun. 2014.

[17] M. Mesbahi and M. Egerstedt, *Graph Theoretic Methods in Multiagent Networks*. Princeton, NJ, USA: Princeton Univ. Press, 2010.

[18] L. Moreau, "Stability of continuous-time distributed consensus algorithms," in *Proc. 43rd IEEE Conf. Decision Control*, 2004, pp. 3998–4003.

[19] L. Moreau, "Stability of multiagent systems with time-dependent communication links," *IEEE Trans. Autom. Control*, vol. 50, no. 2, pp. 169–182, Feb. 2005.

[20] R. Olfati-Saber and R. M. Murray, "Consensus problems in networks with switching topology and time-delays," *IEEE Trans. Autom. Control*, vol. 49, no. 09, pp. 1520–1533, Sep. 2004.

[21] Ed. Ott, *Chaos in Dynamical Systems*. Cambridge, U.K.: Cambridge Univ. Press, 2002.

[22] H. Qing and W. M. Haddad, "Distributed nonlinear control algorithms for network consensus," *Automatica*, vol. 44, no. 9, pp. 2375–2381, 2008.

[23] Motsch S. and Tadmor E, "Heterophilious dynamics enhances consensus," *SIAM: Rev.*, vol. 56, no. 4, pp. 577–621, 2014.

[24] E. Seneta, *Non-Negative Matrices and Markov Chains*. New York, NY, USA: Springer, 2006.

[25] C. Somarakis and J. S. Baras, "Delay-independent convergence for linear consensus networks with applications to non-linear flocking systems," in *Proc. 12th IFAC Workshop Time Delay Syst.*, Ann Arbor, MI, USA, Jun. 2015, pp. 159–164.

[26] C. Somarakis and J. S. Baras, "Non-linear flocking networks with collision avoidance," in *Proc. 34th Chin. Control Conf.*, Jul. 2015, pp. 7633–7638.

[27] C. Somarakis and J. S. Baras, "A simple proof of the continuous time linear consensus problem with applications in non-linear flocking networks," in *Proc. 14th Eur. Control Conf.*, Linz, Austria, Jul. 2015, pp. 1546–1553.

[28] C. Somarakis, R. Hadadi, and J. S. Baras, "Dynamics of collectives with opinionated agents: The case of scrambling connectivity," in *Proc. Eur. Control Conf.*, 2016, pp. 1445–1450.

[29] C. Somarakis, E. Paraskevas, J. S. Baras, and N. Motee, "Synchronization and collision avoidance in non-linear flocking networks of autonomous agents," in *Proc. 24th Mediterranean Conf. Control Autom.*, Jun. 2016, pp. 1089–1094.

[30] C. Sparrow, *The Lorenz Equations: Bifurcations, Chaos, and Strange Attractors*. New York, NY, USA: Springer, 1982.

[31] J. Tsitsiklis, D. Bertsekas, and M. Athans, "Distributed asynchronous deterministic and stochastic gradient optimization algorithms," *IEEE Trans. Autom. Control*, vol. AC-31, no. 9, pp. 803–812, Sep. 1986.

[32] Y. Wenwu, C. Guanrong, and C. Ming, "Some necessary and sufficient conditions for second-order consensus in multi-agent dynamical systems," *Automatica*, vol. 46, pp. 1089–1095, 2010.

[33] Peter Wieland, Rodolphe Sepulchre, and Frank Allgwer, "An internal model principle is necessary and sufficient for linear output synchronization," *Automatica*, vol. 47, no. 5, pp. 1068–1074, 2011.

[34] Chai Wah Wu, *Synchronization in Coupled Chaotic Circuits and Systems*. Singapore: World Scientific, 2002.

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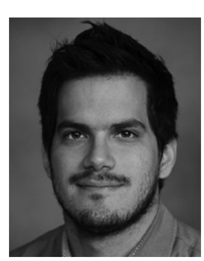

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